On Surrender and Default Risks

Olivier Le Courtois[1] Hidetoshi Nakagawa[2]

[1]EM Lyon, France

[2]Hitotsubashi ICS

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The original paper will be published in *Mathematical Finance*.
Some of the figures displayed below are created for the presentation and are not contained in our original article.

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3. Fund assets and default threshold values
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Qualitative Description of the Problem

- Modeling default risk as well as surrender risk of an asset management entity such as an investment trust fund or an insurance company that issues participating contracts[1].
- Consider the entity that has the following characteristics:
  - The entity collects the resources at time 0 equally from not a few individuals to make a fund pool. In return for the fund, the entity gives some contingent claim to them.
  - The entity can trade a non-defaultable risky asset in the financial market. The entity starts to invest its assets at time 0 and liquidates them at maturity or when a default condition is satisfied.
  - The entity admits no additional funds on the way while early-withdrawals, or surrenders in units of one share of the fund pool are allowed.

[1] 有配当保険契約

Qualitative Description of the Problem

- (Cont'd.) Consider the entity that has the following characteristics:
  - Whenever surrenders happen, the entity cashes out some amount from the fund pool, and then pays off a part of the cash to the participant who surrenders according to the predetermined rule. The remaining cash is withheld as a management cost.
  - The entity enters "default" when the value of the assets drops below a threshold process. Upon default, the company is liquidated and the proceeds are equally distributed according to the contract terms to the remaining participants.
  - Default can be caused due to not only market risk but also surrender risk. In other words, successive surrenders may cause a rapid decrease in the value of the fund pool.
- The main problem is **to achieve the quantitative relationship between the value of the assets and each contingent claim value in consideration of the possibility of default as well as surrender.**

Framework

Modeling default risk and surrender risk

- Default risk modeling is based on a so-called **structural approach**.
  - Default time is defined as some first-passage time of the total asset value.
  - The total asset value jumps downward whenever surrenders happen.
- Surrender risk modeling is based on a so-called **reduced-form approach**.
  - The number of surrenders is specified as a counting process such as Cox process, although some literature studies surrender risk like a kind of American option.
  - In other words, we suppose the policyholders are not completely rational.

Methods and results

- Study the SDEs followed by both the total assets and the asset for each participant.
  - The solutions are obtained via the results for linear SDE in the class of semimartingales.
- Discuss the risk-neutral valuation of the whole contingent claim. For the purpose, it is decomposed into:
  1. The payoff at maturity if no default
  2. The payoff upon default
  3. The accumulated payoffs for surrenders
- Consider how much is needed as equity to make up the fund pool in addition to the amount collected from the participants.
- Show some numerical examples.

Bibliography

- Participating Contracts
  - Brennan and Schwartz [JOFE, 1976]
  - Briys and de Varenne [Geneva Papers, 1994],[JRI, 1997],[Wiley, 2001]
  - Grosen and Jørgensen [Insurance:ME, 2000]
  - Bacinello [ASTIN Bulletin, 2001], [NAAJ, 2003]
  - Andreatta and Corradin [2003]
  - Bernard, Le Courtois and Quittard-Pinon [NAAJ, 2006]
- Prepayment risk of mortgages — similar to surrender risk
  - Schwartz and Torous [JOF, 1989]
  - Nakagawa and Shouda [APFM, 2004]

Mathematical description of the model

- Consider an asset management entity specified by the following items.
  1. Initial fund structure
  2. Fund management
  3. Surrenders
  4. Liquidation before maturity
  5. Liquidation at maturity

General setting / Initial fund structure

- $(\Omega, \mathcal{A}, P)$: a complete probability space
- Q: a (not necessarily unique) risk-neutral probability measure equivalent to P.
- r: the default-free instantaneous interest rate process

(Initial fund structure)

- *The fund pool initially consists of the resources from $I_0 (\in \mathbb{N})$ participants.*
- *Each participant pays $\overline{V}_0$ at time zero and receives a contingent claim in return, hence $I_0 \overline{V}_0$ is the total amount of the fund pool at time zero.*
- *If necessary, additional equity capital E_0 is supposed to be financed.*

Fund management

(Fund management)

- $T \in (0, \infty)$: *the maturity of investment.*
- $(\mathcal{F}_t)$: *the reference filtration without surrender information; e.g. let W_t be a $(P, (\mathcal{F}_t))$-standard Brownian motion and define as*

$$\mathcal{F}_t = \sigma\{r_s, W_s \mid s \leq t\}.$$

- A_t: *the market value process of the whole fund pool invested in the risky asset market. In the case of no liquidation and no surrenders, under P,*

$$dA_t = A_t \left(\mu_t dt + \sigma_t dW_t\right),$$

where μ_t and σ_t are $(\mathcal{F}_t)$-adapted processes satisfying some technical conditions so as to ensure the existence of solution.

- L_t: *the threshold process to trigger the liquidation of the fund even before the maturity.*
- $\tau_d = \inf\{t \in (0, T] \mid A_t \leq L_t\}$: *the liquidation time*

Surrenders

(Surrenders)

- τ^i $(i = 1, \cdots, I_0)$: *the surrender time of the* $\mathbf{i}^{th}$ *participant.*
  - *We will not assume that the* τ^i*'s are* $(\mathcal{F}_t)$*-stopping times, which amounts to relying on reduced-form credit risk valuation.*
- $N_t = \sum_{i=1}^{I_0} N_t^i$ *where* $N_t^i = 1_{\{\tau^i \le t\}}$.
  - N_t *can be regarded as a Cox process by specifying a surrender intensity.*
- $\mathcal{H}_t = \sigma\{N_s \,|\, s \le t\}, \qquad \mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t.$
- φ_t: *a* $(\mathcal{G}_t)$*-adapted càdlàg positive process that stands for the withdrawal from the fund pool if a surrender happens at time* t.
- *The dynamics under* P *of the market value process* A *of the whole fund*

$$dA_t = A_{t-}(\mu_t dt + \sigma_t dW_t) - \varphi_{t-} dN_t = A_{t-}\left(\mu_t dt + \sigma_t dW_t - \frac{\varphi_{t-}}{A_{t-}} dN_t\right),$$

where we set $Y_{t-} := \lim_{s \uparrow t} Y_s$ *for a generic process* Y *if this limit exists.*

Surrenders

(Surrenders (cont'd))

- $\bar{A}_t := \dfrac{A_t}{I_0 - N_t}$: the market value process per a single participant, where $I_0 - N_t$ means the number at time t of the remaining participants in the fund pool.
- $\bar{L}_t := \dfrac{L_t}{I_0 - N_t}$: the threshold process for a single participant
  - $\bar{L}_t$ can be viewed as the discount value of the minimum amount guaranteed to the participant who has not surrendered until maturity T .
- Hereafter we assume that the threshold process L_t is specified by

$$dL_t = L_{t-}\left(\rho_g dt - \frac{dN_t}{I_0 - N_{t-}}\right), \tag{1}$$

where $L_0 = I_0 \bar{L}_T e^{-\rho_g T}$, and ρ_g is a constant that means the guaranteed rate.

  - L_t can be considered as the entity's total discounted debt, so surrender leads to reducing the total debt of the entity.

Explanatory illustration: Entity's asset-liability

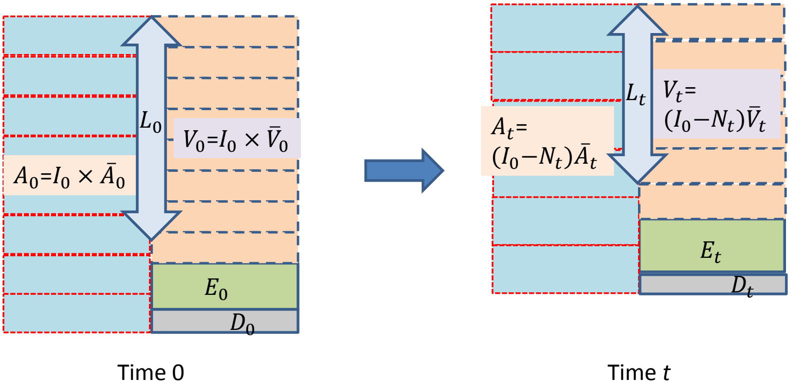

Figure: An illustration of structure and dynamics of the entity's B/S

Explanatory illustration: Surrenders

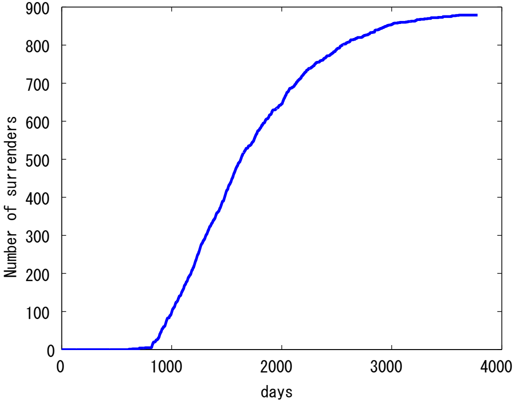

Figure: A sample path of N_t, the number of surrenders

Illustration: Dynamics of the total assets

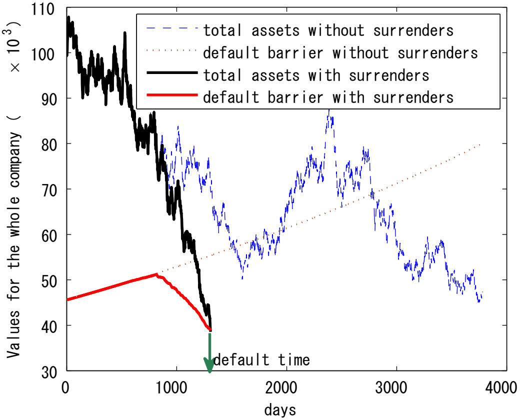

Figure: Sample paths of A_t and L_t, the total assets and the default barrier for the asset management entity

Illustration: Dynamics of the assets per participant

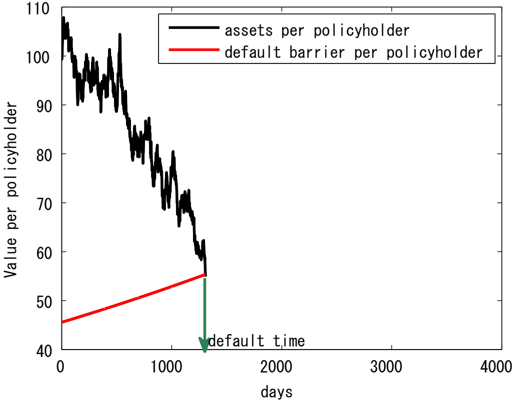

Figure: Sample paths of $\bar{A}_t$ and $\bar{L}_t$, the assets and the default barrier for each participant

Payoffs to the participants

(At surrender)

- $\bar{F}^S(\bar{A}_{t-}, \bar{L}_{t-})$: *the payoff to the participant who surrenders at time* t
  - $\bar{F}^S$ *is a measurable positive function of two variables (or one variable in some instances)*
  - *It is natural to presume* $\bar{L}_{t-} \leq \bar{F}^S(\bar{A}_{t-}, \bar{L}_{t-}) \leq \varphi_{t-}$.

(Liquidation before maturity)

- $\bar{F}^D(\bar{A}_{\tau_d})$ *: the contingent payoff at the liquidation time* $\tau_d(\leq T)$, *where* $\bar{F}^D$ *is a measurable positive function.*
  - $\bar{F}^D(\bar{A}_{\tau_d}) \leq \bar{A}_{\tau_d}$ *should be satisfied. (Typically, we suppose* $\bar{F}^D(a) = a$.*)*

(Liquidation at maturity)

- $\bar{F}(\bar{A}_T, \bar{L}_T)$ *: the contingent payoff at the maturity can be generally, where* $\bar{F}$ *is a measurable positive function of two variables (or one variable in some instances).*
  - $\bar{L}_T \leq \bar{F}(\bar{A}_T, \bar{L}_T) \leq \bar{A}_T$ *should be satisfied.*

A typical payoff of participating contract

A type of participating contract that some insurance companies issue in France is defined by the payoff functions as below:

$$\overline{F}(a, \ell) = \ell + \max\{\delta(\alpha \cdot a - \ell), 0\}, \quad \textit{for some } \alpha, \delta \in (0, 1].$$

The payoff function of $\overline{F}(a, \ell)$ is similar to the payoff of a participating contract discussed in Briys and de Varenne (1994,1997,2001), and Bernard, Le Courtois and Quittard-Pinon (2005).

If one holder of the participating contract is rational, he or she can obtain the optimal surrender time via American option valuation method? ... Not always! Because $\overline{A}_t$ may be dependent on $\mathcal{G}_t$ and the filtration $(\mathcal{G}_t)$ is regarded as the information not for the contract holders but for the insurance companies.

General case

Recall

$$dA_t = A_{t-}(\mu_t dt + \sigma_t dW_t) - \varphi_{t-} dN_t = A_{t-}\left(\mu_t dt + \sigma_t dW_t - \frac{\varphi_{t-}}{A_{t-}} dN_t\right). \qquad (2)$$

Lemma 3.1

The unique solution to (2) is given by

$$A_t = A_0 \exp\left(\int_0^t \mu_u du\right) \mathcal{E}(\sigma \cdot W)_t - \int_0^t \exp\left(\int_s^t \mu_u du\right) \frac{\mathcal{E}(\sigma \cdot W)_t}{\mathcal{E}(\sigma \cdot W)_s} \varphi_{s-} dN_s, \qquad (3)$$

where

$$\mathcal{E}(\sigma \cdot W)_t = \exp\left(-\int_0^t \frac{\sigma_u^2}{2} du + \int_0^t \sigma_u dW_u\right).$$

Note that considering under Q, μ and W are replaced with r and W^Q respectively, where W_t^Q is a Q-standard Brownian motion given by

$$dW_t^Q := dW_t + \frac{\mu_t - r_t}{\sigma_t} dt.$$

General case

Dividing formula (3) by the number of remaining participants, we can write on $\{I_0 > N_t\}$:

$$\bar{A}_t = \frac{A_0}{I_0 - N_t} \exp\left(\int_0^t \mu_u du\right) \mathcal{E}(\sigma \cdot W)_t - \frac{1}{I_0 - N_t} \int_0^t \exp\left(\int_s^t \mu_u du\right) \frac{\mathcal{E}(\sigma \cdot W)_t}{\mathcal{E}(\sigma \cdot W)_s} \varphi_{s-} dN_s.$$

From there, we have the following lemma.

Lemma 3.2

$\bar{A}$ satisfies the next stochastic differential equation on $\{I_0 > N_t\}$:

$$d\bar{A}_t = \bar{A}_{t-}\left(\mu_t dt + \sigma_t dW_t\right) + \frac{\bar{A}_{t-} - \varphi_{t-}}{I_0 - N_t} dN_t.$$

$\tau_d = \inf\{t > 0 \mid \bar{A}_t \leq \bar{L}_t\}$, but τ_d is not an $(\mathcal{F}_t)$-predictable stopping time.
When $\bar{A}_{t-} - \varphi_{t-} < 0$, successive surrenders may cause default due to acceleration of the asset value reduction.

A special case

Suppose $\varphi_t = \bar{A}_t$.

Lemma 3.3

The solution to (2) is given by

$$A_t = A_0 \exp\left(\int_0^t \mu_u du\right) \mathcal{E}(\sigma \cdot W)_t \frac{I_0 - N_t}{I_0}.$$

Corollary 3.4

The solution to (1) is given by

$$L_t = L_0 e^{\rho_g t} \frac{I_0 - N_t}{I_0} = (I_0 - N_t) \bar{L}_T e^{-\rho_g (T-t)}.$$

It follows from this lemma that τ_d can be represented as $\inf\{t > 0 \mid \bar{A}_t \leq \bar{L}_t\}$, so τ_d is an $(\mathcal{F}_t)$-predictable stopping time.
Default is not directly caused by any surrender, so the problem is just reduced to the first-passage time valuation like Black-Cox model.

General formulae of contingent claim valuation

Definition 4.1 (Liabilities w.r.t. policyholders)

The total initial value of the participating contracts is given by $V_0 := V_0^1 + V_0^2 + V_0^3$.

- The value of payoff at the maturity to the participant who has not surrendered unless default:

$$V_0^1 := E^Q\left[(I_0 - N_T)(1 - M_T)e^{-\int_0^T r_u du}\bar{F}(\bar{A}_T, \bar{L}_T)\right].$$

- The value of the payoff at default to the participant who has not yet surrendered:

$$V_0^2 := E^Q\left[\int_0^T (I_0 - N_{s-})e^{-\int_0^s r_u du}\bar{F}^D(\bar{A}_{s-})dM_s\right].$$

- The value of the cumulative payoff to who surrenders before default (default is senior to surrenders):

$$V_0^3 := E^Q\left[\int_0^T (1 - M_s)e^{-\int_0^s r_u du}\bar{F}^S(\bar{A}_{s-}, \bar{L}_{s-})dN_s\right].$$

Here $M_t := \mathbf{1}_{\{\tau_d \leq t\}}$.

Comments

- In fact, our paper ignores some important problems.
  - How to specify the dynamics of N_t, in short, the surrender intensity process?
  - How to estimate the parameters included in the surrender intensity?
  - What is the difference between the surrender intensity under $\boldsymbol{P}$ and that under $\boldsymbol{Q}$?
- If the surrender intensity process is specified, the conditional expectations in the last slide can be represented in a more tractable form.
- What approach is appropriate for specifying the surrender intensity process?
  - For a numerical example below, a common Vasicek-type intensity process (correlated with interest rate and the asset) is supposed for each participant's surrender and Gaussian copula with one correlation parameter is supposed for dependence structure. (Although both are criticized!)
  - Some contagious intensity process that is introduced in the top-down approach of credit risk modeling may be better...

Equity capital necessary for constructing the fund pool

Lemma 4.2

Assume that $\int_0^t \bar{A}_{s-}\sigma_s dW_s^Q$ is a $(Q, \mathcal{G})$-true martingale and that M_t and N_t do not jump simultaneously a.s.
Then we have

$$A_0 = E^Q\left[(I_0 - N_T)(1 - M_T)e^{-\int_0^T r_u du}\bar{A}_T\right] + E^Q\left[\int_0^T (I_0 - N_{s-})e^{-\int_0^s r_u du}\bar{A}_{s-}dM_s\right] + E^Q\left[\int_0^T (1 - M_s)e^{-\int_0^s r_u du}\varphi_{s-}dN_s\right].$$

Equity capital necessary for constructing the fund pool

Proposition 4.4

Assume that the conditions of Lemma 4.2. Moreover if $\bar{F}^S(\bar{A}_{t-}, \bar{L}_{t-}) \leq \varphi_{t-}$ for any $t \in (0, T]$ and if for any $a > 0$,

$$\bar{F}(a, \cdot) \leq a, \bar{F}^D(a) \leq a,$$

then $V_0 \leq A_0$.
In particular, if all the equalities in the above inequality conditions hold, we can easily see that $V_0 = A_0$ is satisfied.

Equity capital necessary for constructing the fund pool

Proof.

Since $V_0 = V_0^1 + V_0^2 + V_0^3$, due to the conditions we have

$$
\begin{aligned}
V_0 \le\ & E^Q\left[(I_0 - N_T)(1 - M_T)e^{-\int_0^T r_u du}\bar{A}_T\right] + E^Q\left[\int_0^T (I_0 - N_{s-})e^{-\int_0^s r_u du}\bar{A}_{s-}dM_s\right] \\
& + E^Q\left[\int_0^T (1 - M_s)e^{-\int_0^s r_u du}\varphi_{s-}dN_s\right] \\
=\ & A_0,
\end{aligned}
$$

where the last equality follows from Lemma 4.2.

□

Valuation of Equity

Proposition 4.5

$A_0 - V_0$ can be decomposed as $A_0 - V_0 = E_0^1 + E_0^2 + D_0$, where

$$E_0^1 := E^Q\left[(I_0 - N_T)(1 - M_T)e^{-\int_0^T r_u du}\left\{\bar{A}_T - \bar{F}(\bar{A}_T, \bar{L}_T)\right\}\right],$$

$$E_0^2 := E^Q\left[\int_0^T (I_0 - N_{s-})e^{-\int_0^s r_u du}\left\{\bar{A}_{s-} - \bar{F}^D(\bar{A}_{s-})\right\} dM_s\right],$$

$$D_0 := E^Q\left[\int_0^T (1 - M_s)e^{-\int_0^s r_u du}\left\{\varphi_{s-} - \bar{F}^S(\bar{A}_{s-}, \bar{L}_s)\right\} dN_s\right].$$

Definition 4.7 (Equity)

Define the market value of equity as the difference between the value of the assets and the sum of the market value of the liabilities with respect to policyholders and of the discounted future management costs, namely as $E_0 := A_0 - V_0 - D_0 = E_0^1 + E_0^2$.

Outline of numerical illustration

- What to compute:
  - Default probability $P(\tau_d \leq T)$ for a fixed T
  - Values of each portion V_0^1, V_0^2, V_0^3 and the total liabilities V_0 as well as the equity value E_0 and the discounted future managing costs D_0 (under Q)
- How to compute:
  - Monte Carlo simulation with 20,000 trials for each case
  - Comparison of the values for changing the following two parameters for surrender risk
    - λ_0: the initial common surrender intensity for each policyholder
    - ρ: the common correlation of Gaussian copula

Specification of participating contracts

- Consider a type of participating contract that some insurance companies issue and that is defined by the payoff functions

$$\bar{F}(a,\ell) = \ell + \max\{\delta(\alpha \cdot a - \ell), 0\}, \bar{F}^D(a) = a, \bar{F}^S(a,\ell) = \ell,$$

  where $\alpha, \delta \in (0,1]$.
  - The specification of $\bar{F}^S(a,\ell) = \ell$ implies that it is relatively inconvenient for the participants to surrender since they receive only at most the single threshold $\bar{L}_t$ even if the fund performance is quite strong.
- Withdrawal from the assets upon surrender: $\varphi_{t-} = \beta \bar{A}_{t-}$ for a constant $\beta > 1$. This means that surrenders jump downwards the market value process $\bar{A}_t$ for each participant who remains

Model specification under P

Under the physical probability P:

- The dynamics of assets and default barrier:

$$
\begin{aligned}
d\bar{A}_t &= \bar{A}_{t-}\left(\mu dt + \sigma_A dW_t\right) + \frac{\bar{A}_{t-} - \beta_{t-}\bar{L}_t}{I_0 - N_t} dN_t \quad \text{if } I_0 > N_t, \\
d\bar{L}_t &= \bar{L}_t \rho_g dt \quad (\bar{L}_t = \bar{L}_0 e^{\rho_g t}), \\
A_t &= \bar{A}_t (I_0 - N_t), \quad L_t = \bar{L}_t (I_0 - N_t).
\end{aligned}
$$

- The interest rate: $dr_t = a_r\left(b_r - r_t\right) dt + \sigma_r\, dZ_t$
- The common surrender intensity: $d\lambda_t = a_\lambda\left(b_\lambda - \lambda_t\right) dt + \sigma_\lambda\, dX_t$
- Assume W_t, Z_t and X_t are $(\mathcal{F}_t)$-standard Brownian motions and

$$
\rho_{Ar} dt := dW_t dZ_t,\ \rho_{A\lambda} dt := dW_t dX_t,\ \rho_{r\lambda} dt := dZ_t dX_t.
$$

Model specification under Q

Under the risk-neutral probability Q:

- The drift of the assets is given by the interest rate r_t.
- The interest rate dynamics is completely the same as that under P.
- The common surrender intensity is given by $\lambda_t^Q = \zeta \lambda_t$ where ζ is a positive constant. (This ζ may be interpreted as a sort of risk premium for surrender risk.)
- The correlations are invariant by the change of measure.

Surrender times simulation

- Generate $\varepsilon^1, \varepsilon^2, \cdots, \varepsilon^{I_0}$ satisfying
    - Each ε^i is an exponential random variable with intensity 1.
    - Dependence is given by Gaussian copula with common parameter ρ.
- Compute $\tau^1, \tau^2, \cdots, \tau^{I_0}$ by

$$\tau^i := \inf\left\{ t > 0 \middle| \int_0^t \lambda_s ds \geq \varepsilon^i \right\}.$$

- Note that the intensities are simulated in such a way that they never become negative (although the probability is very small) by making them rebound at zero, if necessary.

Table: Case study parameters on the initial conditions of the fund and the contingent payoff

A_0	L_0	ρ_g	δ	I_0	T	α	β
100	80	0.0375	0.9	1000	15	$\mathbf{0.8}$	$\mathbf{1.05}$

Remark: We discretized time into $252 \times 15 = 3,780$ steps.

Table: Case study parameters on the other parameters

r_0	b_r	a_r	λ_0	b_λ	a_λ	ζ	μ
0.04	0.055	35	0.05	0.05	30	0.1	0.07

Table: Case study parameters on the volatilities and the correlations

σ_A	σ_r	σ_λ	ρ_{Ar}	$\rho_{A\lambda}$	$\rho_{r\lambda}$	ρ
0.08	0.05	0.05	-0.5	-0.5	0.5	0.1

Ruin Probability w.r.t. Surrender Intensity

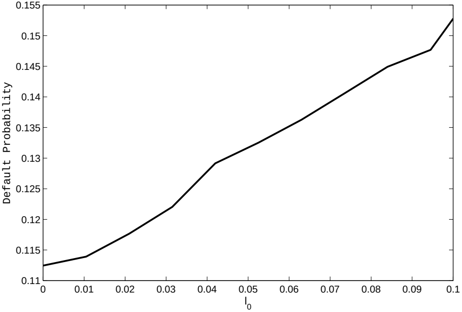

Figure: Comparison of the default probability $P(\tau_d \leq 15)$ for changing the initial surrender intensity λ_0 (under the assumption $\lambda_0 = b_\lambda$) between 0 and 0.1

Default probability w.r.t. correlation ρ

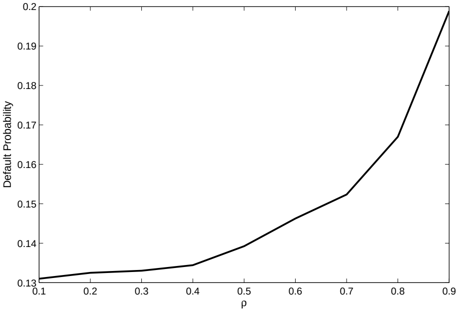

Figure: Comparison of $P(\tau_d \leq 15)$ for changing the common correlation parameter ρ (of a Gaussian copula) between 0.1 and 0.9

Comparison of V_0^1, V_0^2, V_0^3, and V_0 for various λ_0

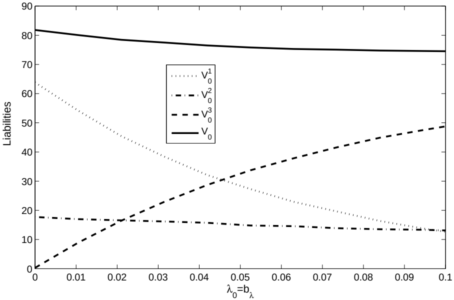

Figure: Comparison of the liabilities V_0^1, V_0^2, V_0^3, and V_0 for various λ_0 values (under the assumption $\lambda_0 = b_\lambda$) between 0 and 0.1

Comparison of V_0^1, V_0^2, V_0^3, and V_0 for various ρ

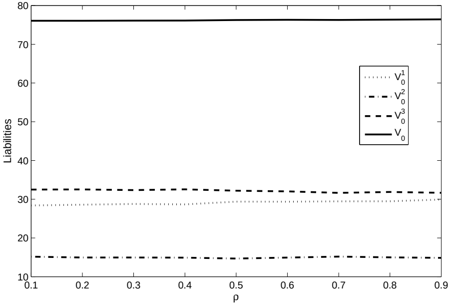

Figure: Comparison of V_0^1, V_0^2, V_0^3, and V_0 for various ρ values between 0.1 and 0.9

Comparison of the equity E_0 and the expected discount total management cost D_0 for various λ_0

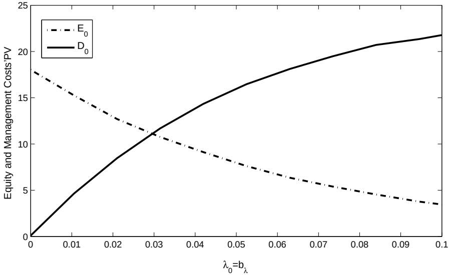

Figure: Comparison of the equity E_0 and the expected discount total management cost D_0 for various λ_0 values (under the assumption $\lambda_0 = b_\lambda$) between 0 and 0.1

Concluding Remarks

- We modeled default risk as well as surrender risk of an asset management entity.
  - Default time: the first-passage time of the total assets to the amount guaranteed at maturity which is discounted by the guaranteed rate
  - Surrender times: a counting process (just specified only for the numerical example)
- We illustrated one tentative numerical example about the default probability and the initial values of components of a typical participating contract.
- There remain many research tasks...